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The blocked composite operators are defined in the one-component Euclidean scalar field theory, and shown to generate a linear transformation of the operators, the operator mixing. This transformation allows us to introduce the parallel transport of the operators along the RG trajectory. The connection on this one-dimensional manifold governs the scale evolution of the operator mixing. It is shown that the solution of the eigenvalue problem of the connection gives the various scaling regimes and the relevant operators there. The relation to perturbative renormalization is also discussed in the framework of the ϕ^3 theory in dimension $d = 6$.

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I. INTRODUCTION

The RG strategy is to follow the evolution of the coupling constants in the observational scale in order to identify the important interactions at different scales. This is achieved by the blocking, the successive lowering of the UV cut-off and the tracing of the resulting blocked, renormalized action [1,2]. There is an alternative method, implicit in this procedure, where the original cut-off and action are kept but the operators are modified to cover less modes while keeping the expectation values fixed. This paper outlines this latter method in a more systematic manner and compares it with the traditional one.

A similar question arises in the renormalization of composite operators where the goal is to render the Green's functions containing the insertion of some local operators which are not in the action finite as the cut-off is removed. The perturbative treatment of this problem proceeds by the introduction of additional counterterms in the action in such a manner that the original Green's functions remain unchanged but those with the insertion of the composite operators turn out to be finite [3–6].

This rather complicated procedure attempts to perform an amazing project, the renormalization of an otherwise non-renormalizable model. This happens because the insertion of the non-renormalizable composite operators in the Green's functions can be achieved by introducing them in the action with a source term and taking the derivative of the logarithm of the partition function with respect to their source. As long as the Green's functions are finite so is the partition function containing non-renormalizable operators! The resolution of this apparent paradox is that the rendering of more Green's functions with the composite operator insertion finite requires more counterterms. At the end we should need infinitely many counterterms to complete the project, just as when we would attempt to remove the cut-off in a non-renormalizable model.

The old-fashioned perturbative proof of renormalizability which is made involved by the tracing of overlapping divergences [6] was simplified enormously by considering a general, non-perturbative renormalization scheme [7]. In fact, the renormalizability turns out to be the absence of the UV Landau pole in the latter which can easily be avoided in the framework of the loop expansion for asymptotically free models. Our introduction of the operator renormalization as an alternative of the blocking appears as a similar, non-perturbative setup to discuss the composite operator renormalization and shows clearly the source of the complications when non-renormalizable operators are considered.

The usual blocking is based on local quantities of the RG flow, such as the beta-functions which are sufficient for the local studies in the vicinity of a given fixed point. But the RG flow of the more realistic models visits several scaling regimes between the UV and the IR regimes and the determination of the set of important parameters may require global methods which can take into account the interplay between different scaling regimes [8]. The operator renormalization includes in a natural manner a global quantity, called sensitivity matrix, needed for such an analysis. In this manner it is better suited for the studies of models with non-trivial IR scaling law as the traditional blocking.

In Sect. II a toy model is presented explaining the origin of operator mixing. Then, the idea of blocking is extended to operators in one-component scalar field theory in Sect. III and in Sect. IV the differential geometric meaning of blocking the operators is clarified. The possibility of finding the scaling operators at an arbitrary scale by means of solving the eigenvalue problem of the sensitivity matrix is discussed in Sect. V. The construction of the blocked operator and the renormalized perturbation expansion (RPE) are compared in Sect. VI. The resolution of the paradoxon of renormalizing irrelevant composite operators in a renormalizable theory is also discussed. The operator mixing matrix is determined for ϕ^3 theory in dimension $d = 6$ in a restricted operator basis in the independent mode approximation (IMA) in two different ways in Sects. VII A 1 and VII A 2. The agreement of these results with the one-loop perturbative ones are shown in Sect. VII B. Finally, the main results of the paper are summarized in Sect. VIII.

II. TOY MODEL FOR OPERATOR MIXING

Let us consider a zero-dimensional model with two degrees of freedom, x ('the low-frequency one') and y ('the high-frequency one'), and the bare action

$$S(x, y) = \frac{1}{2}s_x x^2 + \frac{1}{2}s_y y^2 + \sum_{n=0}^{\infty} g_n(x + y)^n \tag{2.1}$$

with the bare couplings g_n. We are looking for the blocked action $S(x)$ obtained by integrating out the degree of freedom y,

$$e^{-S(x)} = \int dy e^{-S(x,y)}. \tag{2.2}$$

The composite operators of the bare theory are defined as $(x + y)^n$ with $n = 0, 1, 2, \ldots$ According to relation (2.2), the blocked action $S(x)$ as the function of the bare coupling constants g_n can be considered as the generator function for the composite operators for a given value x,

$$\frac{\partial S(x)}{\partial g_n} = \frac{\int dy(x + y)^n e^{-S(x,y)}}{\int dy e^{-S(x,y)}}, \tag{2.3}$$

i.e. the partial derivative of the blocked action w.r.t. one of the bare couplings g_n is equal to the 'high-frequency' average of the corresponding bare composite operator $(x + y)^n$. As a consequence the relation

$$\frac{\int dx \frac{\partial S(x)}{\partial g_n} e^{-S(x)}}{\int dx e^{-S(x)}} = \frac{\int dxdy(x + y)^n e^{-S(x,y)}}{\int dxdy e^{-S(x,y)}} \tag{2.4}$$

holds, i.e. the partial derivative of the blocked action w.r.t. the bare coupling g_n can be interpreted as the blocked operator that provides the same expectation value in the blocked theory as the bare operator $(x + y)^n$ does in the bare theory. Let us introduce the notation $\{x^n\} = \partial S(x)/\partial g_n$ for it. The action $S(x)$ can be expanded in the terms of the base operators as

$$S(x) = \frac{1}{2}s_x x^2 + \sum_{m=0}^{\infty} g'_m x^m \tag{2.5}$$

with the blocked couplings $g'_m = g'_m(g_0, g_1, g_2, \ldots)$ $(m = 0, 1, 2, \ldots)$ being functions of the bare couplings. Using this expansion, the blocked operators can be rewritten in the form:

$$\{x^n\} = \frac{\partial S(x)}{\partial g_n} = \sum_{m=0}^{\infty} x^m S_{mn} \tag{2.6}$$

with the help of the matrix

$$S_{mn} = \frac{\partial g'_m}{\partial g_n}. \tag{2.7}$$

Eq. (2.6) has the simple interpretation that the blocked operator $\{x^n\}$ obtained from the bare operator $(x + y)^n$ by integrating out the degree of freedom y is a linear combination of the base operators x^m with the coefficients given by the operator mixing matrix S_{mn}. Thus, the operator with which one can reproduce the vacuum expectation value of a bare base operator in the blocked theory is the linear combination of all the base operators with arguments projected into the subspace of the degree of freedom x left over.

Above we chose base operators in the bare theory and searched for operators in the blocked theory that reproduce their vacuum expectation values. Also the opposite question can be formulated: we choose a basis of operators in the blocked theory, x^n with $n = 0, 1, 2, \ldots$ and search for the operators $[(x+y)^n]$ of the bare theory that reproduce their expectation values, i.e. for which

$$\frac{\int dxdy[(x+y)^n]e^{-S(x,y)}}{\int dxdye^{-S(x,y)}} = \frac{\int dxx^n e^{-S(x)}}{\int dxe^{-S(x)}} \tag{2.8}$$

assuming that the bare theory underlying the blocked one is known. Looking for the bare operators in the general form $[(x+y)^n] = \sum_{m=0}^{\infty} \alpha_m^{(n)}(x+y)^m$, and using Eqs. (2.4) and (2.6), Eq. (2.8) leads to the relation

$$\sum_{m=0}^{\infty} x^{n'} S_{n'm}\alpha_m^{(n)} = x^n \tag{2.9}$$

and one finds that the matrix $\alpha_m^{(n)} = (S^{-1})_{mn}$ is the inverse of the operator mixing matrix given by Eq. (2.6). Thus, the bare operators looked for are given by

$$[(x+y)^n] = \sum_{m=0}^{\infty}(x+y)^m (S^{-1})_{mn}. \tag{2.10}$$

We see again, that the bare operators reproducing the expectation values of given base operators of the blocked theory in the bare one are the linear combinations of the base operators.

It is instructive to generalize the toy model for three variables,

$$S(x,y,z) = \frac{1}{2}s_x x^2 + \frac{1}{2}s_y y^2 + \frac{1}{2}s_z z^2 + \sum_{n=0}^{\infty} g_n (x+y+z)^n. \tag{2.11}$$

The blocking gives rise the chain of effective theories,

$$e^{-S(x,y)} = \int dze^{-S(x,y,z)}, \qquad e^{-S(x)} = \int dye^{-S(x,y)}, \tag{2.12}$$

and operator mixing

$$\{x^n\}_z = \frac{\partial S(x,y,z)}{\partial g_n} = (x+y+z)^n, \qquad \{x^n\}_y = \frac{\partial S(x,y)}{\partial g_n} = -\frac{\frac{\partial}{\partial g_n}\int dze^{-S(x,y,z)}}{\int dze^{-S(x,y,z)}},$$
$$\{x^n\}_x = \frac{\partial S(x)}{\partial g_n} = -\frac{\frac{\partial}{\partial g_n}\int dydze^{-S(x,y,z)}}{\int dydze^{-S(x,y,z)}}. \tag{2.13}$$

These relations yield

$$\{x^n\}_y = \frac{\int dz\{x^n\}_z e^{-S(x,y,z)}}{\int dze^{-S(x,y,z)}}, \qquad \{x^n\}_x = \frac{\int dy\{x^n\}_y e^{-S(x,y)}}{\int dye^{-S(x,y)}}, \tag{2.14}$$

indicating that the evolution of the operators comes from the elimination of the field variable in their definition. One can compute the expectation value of $\{x^n\}$ at any level,

$$\frac{\int dxdydz\{x^n\}_z e^{-S(x,y,z)}}{\int dxdydze^{-S(x,y,z)}} = \frac{\int dxdy\{x^n\}_y e^{-S(x,y)}}{\int dxdye^{-S(x,y)}} = \frac{\int dx\{x^n\}_x e^{-S(x)}}{\int dxe^{-S(x)}}. \tag{2.15}$$

In other words, the cut-off dependence of the renormalized operators is introduced in such a manner that the expectation values can be recovered for an arbitrary value of the cut-off.

We learned on the above discussed toy-model that operator mixing is an immediate consequence of keeping the expectation values unchanged under integrating out degrees of freedom.

III. OPERATOR MIXING

Here we define blocked operators in quantum field theory following the line illustrated by the zero-dimensional toy-model of the previous section.

Let us consider the theory given by the bare action

$$S_\Lambda[\phi] = \int dx \sum_n G_n(x,\Lambda) O_n(\phi(x)), \tag{3.1}$$

where $O_n(\phi(x))$ represents a complete set of local operators (function of $\phi(x)$ and its space-time derivatives) coupled to the external sources $G_n(x,\Lambda)$. The scalar field $\phi(x) = \varphi_k(x) + \chi_k(x)$ is decomposed into a low-frequency part

$$\varphi_k(x) = \sum_{|p|\le k} \phi_p e^{ipx} \tag{3.2}$$

and a high-frequency part

$$\chi_k(x) = \sum_{|p|\in[k,\Lambda]} \phi_p e^{ipx} \tag{3.3}$$

with the UV cut-off Λ and the sharp moving cut-off k. Here inhomogeneous external sources have been introduced for later convenience and assumed that their zero modes $g_n(\Lambda)$ are separated,

$$G_n(x,\Lambda) = g_n(x,\Lambda) + g_n(\Lambda). \tag{3.4}$$

Notice that the variables n and x of the operator $O_n(\phi(x))$ identify a member of the complete set. One can simplify the expressions by introducing a single index $\tilde{n}$ for the pair (n,x) and $\sum_{\tilde{n}}$ for the integral and sum $\sum_n \int dx$, e.g.

$$S_\Lambda[\phi] = \sum_{\tilde{n}} G_{\tilde{n}}(\Lambda) O_{\tilde{n}}(\phi). \tag{3.5}$$

It is required that the partition function of the blocked theory Z_k and that of the bare one Z_Λ are identical:

$$\frac{\int \mathcal{D}\varphi_k \mathcal{D}\chi_k e^{-S_\Lambda[\varphi_k+\chi_k]}}{\int \mathcal{D}\varphi_k \mathcal{D}\chi_k e^{-S_\Lambda[\varphi_k+\chi_k]}\Big|_{g_n(x,\Lambda)=0}} = \frac{\int \mathcal{D}\varphi_k e^{-S_k[\varphi_k]}}{\int \mathcal{D}\varphi_k e^{-S_k[\varphi_k]}\Big|_{g_n(x,k)=0}} \tag{3.6}$$

that leads to the definition of the blocked action $S_k[\varphi_k]$ up to a constant,

$$e^{-S_k[\varphi_k]} = \int \mathcal{D}\chi_k e^{-S_\Lambda[\varphi_k+\chi_k]}. \tag{3.7}$$

As shown in [2] the blocked action satisfies the Wegner-Houghton equation

$$\frac{\partial S_k[\varphi_k]}{\partial k} = -\lim_{\delta k\to 0} \frac{1}{2\delta k} Tr \ln \frac{\delta^2 S_k}{\delta\varphi_k \delta\varphi_k} \tag{3.8}$$

where the trace is taken over the functional space with Fourier modes $k - \delta k < p < k$. The blocked action can be expanded in the base operators as

$$S_k[\varphi_k] = \sum_{\tilde{n}} G_{\tilde{n}}(k) O_{\tilde{n}}(\varphi_k) \tag{3.9}$$

and (3.8) rewritten in the form of a coupled set of differential equations

$$k\partial_k G_{\tilde{n}}(k) = \beta_{\tilde{n}}(k,G) \tag{3.10}$$

for the blocked external sources $G_{\tilde{n}}(k)$. The right hand sides of these equations represent explicit expressions for the beta-functions as functions of the blocked sources.

Let us now turn to the definition of blocked operators. From Eq. (3.7), one gets after functional differentiation

$$\frac{\delta S_k[\varphi_k]}{\delta G_{\tilde{n}}(\Lambda)} = \frac{\int \mathcal{D}\chi_k O_{\tilde{n}}(\varphi_k + \chi_k) e^{-S_\Lambda[\varphi_k+\chi_k]}}{\int \mathcal{D}\chi_k e^{-S_\Lambda[\varphi_k+\chi_k]}}, \tag{3.11}$$

i.e. the functional derivative of the blocked action w.r.t. any of the bare sources $G_{\tilde{n}}(\Lambda)$ is equal to the high-frequency average of the corresponding bare operator $O_{\tilde{n}}(\varphi_k + \chi_k)$. Thus the functional derivative $\delta S_k[\varphi_k]/\,\delta G_{\tilde{n}}(\Lambda)$ reproduces the expectation value of the bare operator $O_{\tilde{n}}(\varphi_k + \chi_k)$,

$$\frac{\int \mathcal{D}\varphi_k \frac{\delta S_k[\varphi_k]}{\delta G_{\tilde{n}}(\Lambda)} e^{-S_k[\varphi_k]}}{\int \mathcal{D}\varphi_k e^{-S_k[\varphi_k]}} = \frac{\int \mathcal{D}\varphi_k \mathcal{D}\chi_k O_{\tilde{n}}(\varphi_k + \chi_k) e^{-S_\Lambda[\varphi_k+\chi_k]}}{\int \mathcal{D}\varphi_k \mathcal{D}\chi_k e^{-S_\Lambda[\varphi_k+\chi_k]}}. \tag{3.12}$$

As a result the functional derivative of the blocked action w.r.t. one of the bare external sources $G_{\tilde{n}}(\Lambda)$ can be interpreted as the operator obtained by blocking from the base operator $O_{\tilde{n}}(\varphi_k + \chi_k)$. Let us introduce the notation

$$\{O_{\tilde{n}}(\varphi_k)\}_k = \frac{\delta S_k[\varphi_k]}{\delta G_{\tilde{n}}(\Lambda)} \tag{3.13}$$

for the corresponding blocked operator. Making use of expansion (3.9) of the blocked action, the operator mixing

$$\{O_{\tilde{n}}(\varphi_k)\}_k = \sum_{\tilde{m}} \frac{\delta G_{\tilde{m}}(k)}{\delta G_{\tilde{n}}(\Lambda)} \frac{\delta S_k[\varphi_k]}{\delta G_{\tilde{m}}(k)} = \sum_{\tilde{m}} O_{\tilde{m}}(\varphi_k) S_{\tilde{m}\tilde{n}}(k,\Lambda) \tag{3.14}$$

can be given in terms of the sensitivity matrix

$$S_{\tilde{m}\tilde{n}}(k,\Lambda) = \frac{\delta G_{\tilde{m}}(k)}{\delta G_{\tilde{n}}(\Lambda)}, \tag{3.15}$$

which shows the dependence of the blocked coupling constants on the initial values of the RG flow. As such it is a global feature of the RG trajectory.

Conversely, if one chooses the base operators $O_{\tilde{n}}(\varphi_k)$ at the scale k then one can search for the bare operator $[O_{\tilde{n}}(\varphi_k + \chi_k)]_k$ that reproduces the vacuum expectation value of $O_{\tilde{n}}(\varphi_k)$. An argument similar to that for the toy-model in the previous section leads to

$$[O_{\tilde{n}}(\varphi_k + \chi_k)]_k = \sum_{\tilde{m}} O_{\tilde{m}}(\varphi_k + \chi_k)(S^{-1}(k,\Lambda))_{\tilde{m}\tilde{n}}. \tag{3.16}$$

Now we pass the RG trajectory in the opposite direction (from small k to large k values) when asking for the bare operator that reproduces the same vacuum expectation value as a given operator at the scale of small k. Therefore, the answer is given in terms of the inverse of the matrix $S_{\tilde{m}\tilde{n}}(k,\Lambda)$ describing the mixing of operators if the RG trajectory is passed from large k values towards the small ones as it happened when defining the blocked operators $\{O_{\tilde{n}}(\varphi_k)\}_k$. Notice that the relations (3.14) and (3.16) represent operator equations since Eq. (3.12) remains valid after the insertion of any other operator in the path integral of the numerator at both sides.

It turns out to be useful to derive partial differential equations for the k dependence of the operator mixing matrix. Using

$$G_{\tilde{n}}(k - \delta k) = G_{\tilde{n}}(k) - \frac{\delta k}{k} \beta_{\tilde{n}}(k, G) \tag{3.17}$$

for an arbitrary infinitesimal change δk of the scale k, one can write

$$S_{\tilde{m}\tilde{n}}(k - \delta k) = \frac{\delta G_{\tilde{m}}(k-\delta k)}{\delta G_{\tilde{n}}(\Lambda)} = \sum_{\tilde{\ell}} \frac{\delta G_{\tilde{m}}(k-\delta k)}{\delta G_{\tilde{\ell}}(k)} \frac{\delta G_{\tilde{\ell}}(k)}{\delta G_{\tilde{n}}(\Lambda)} = \sum_{\tilde{\ell}} \left[\delta_{\tilde{m}\tilde{\ell}} - \frac{\delta k}{k} \frac{\delta \beta_{\tilde{m}}(k,G)}{\delta G_{\tilde{\ell}}(k)}\right] S_{\tilde{\ell}\tilde{n}}(k). \tag{3.18}$$

Subtracting Eq. (3.15), dividing by δk and taking the limit $\delta k \to 0$, we find the set of coupled differential equations for the elements of the operator mixing matrix:

$$k\partial_k S_{\tilde{m}\tilde{n}}(k,\Lambda) = \sum_{\tilde{\ell}} \frac{\delta \beta_{\tilde{m}}(k)}{\delta G_{\tilde{\ell}}(k)} S_{\tilde{\ell}\tilde{n}}(k,\Lambda). \tag{3.19}$$

The scale dependence of the operator mixing matrix is governed by the matrix

$$k\Gamma_{\tilde{n}\tilde{m}}(k) = \frac{\delta \beta_{\tilde{n}}(k)}{\delta G_{\tilde{m}}(k)} \tag{3.20}$$

determining the change of the slope of the RG trajectory due to the variation of the actual point of the parameter space it passes through. The matrix $\Gamma_{\tilde{n}\tilde{m}}(k)$ is a local feature of the RG trajectory on the contrary to the sensitivity matrix.

Now we show that the operator mixing due to blocking can be interpreted as the parallel transport of operators along the RG trajectory with the connection $\Gamma_{\tilde{n}\tilde{m}}(k)$.

It is the main idea behind operator renormalization that operators are searched for that reproduce the same vacuum expectation value at different scales,

$$\frac{\int \mathcal{D}\phi_{k'}\mathcal{D}\chi_{k'}O_{\tilde{n}}(\varphi_{k'}+\chi_{k'})e^{-S_{k'}[\varphi_{k'}+\chi_{k'}]}}{\int \mathcal{D}\phi_{k'}\mathcal{D}\chi_{k'}e^{-S_{k'}[\varphi_{k'}+\chi_{k'}]}} = \frac{\int \mathcal{D}\phi_{k}\mathcal{D}\chi_{k}O_{\tilde{n}}(\varphi_{k}+\chi_{k})e^{-S_{k}[\varphi_{k}+\chi_{k}]}}{\int \mathcal{D}\phi_{k}\mathcal{D}\chi_{k}e^{-S_{k}[\varphi_{k}+\chi_{k}]}}. \tag{4.1}$$

This relation follows from observing that the right hand side of Eq. (3.12), the expectation value of a given bare operator of the complete cut-off theory, is independent of the choice of k, the way the modes are split into the UV and IR classes. Eq. (4.1) allows us to identify the effect of the changing of the cut-off in the renormalized operator,

$$\{O_{\tilde{n}}(\varphi_{k'})\}_{k'} = \frac{\int \mathcal{D}\xi\{O_{\tilde{n}}(\varphi_{k'}+\xi)\}_k e^{-S_k[\varphi_{k'}+\xi]}}{\int \mathcal{D}\xi e^{-S_k[\varphi_{k'}+\xi]}}, \tag{4.2}$$

for $k' < k$. The integration $\mathcal{D}\xi$ extends over the modes with $k' < |p| < k$. The operator $\{O_{\tilde{n}}(\varphi_{k'})\}_{k'}$ is a linear superposition of the base operators $O_{\tilde{n}}(\varphi_{k'})$ as indicated in Eq. (3.14).

This result suggests a differential geometric interpretation of the operator mixing, its identification with a certain parallel transport. In particular, a k dependent operator O_k will be said to be parallel transported in the scale k,

$$O_{k'} = P_{k\to k'}O_k \tag{4.3}$$

if its insertion into any expectation value yields a k-independent result, $\partial_k\langle O_k\rangle = 0$. The mapping $P_{k\to k'}$ of the operators is obviously linear.

Since the operator mixing is linear according to Eq. (3.14), the parallel transport of an operator in the scale, i.e. the parallel transported operator O_k can be characterized by introducing the covariant derivative of the operators,

$$D_kO_k = (\partial_k - \Gamma)O_k \tag{4.4}$$

in such a manner that $D_kO_k = 0$ for the parallel transport. Eq. (3.19) suggests the identification of the connection with the matrix Γ of (3.20).

The formal definition of the covariant derivative is the following. The scale dependence in the expectation value $\langle O_k\rangle$ comes from two different sources: from the explicit k dependence of the operator and the implicit k dependence generated by the path integration, by taking the expectation value. The operator mixing ballances them. We introduce the covariant derivative by the relation

$$\partial_k\langle O_k\rangle = \langle D_kO_k\rangle, \tag{4.5}$$

requiring that the operator mixing generated by the connection amounts to the implicit k dependence of the expectation value. Once the covariant derivative is known the parallel transport can be reconstructed as

$$P_{k\to k'} = \mathcal{P}e^{\int_k^{k'} dk''\Gamma(k'')} \tag{4.6}$$

where $\mathcal{P}$ stands for the ordering according to the parameter k''.

It is obvious that the connection Γ is vanishing in the basis $\{O_{\tilde{n}}\}_k$,

$$\partial_k\langle\sum_{\tilde{n}} c_{\tilde{n}}(k)\{O_{\tilde{n}}\}_k\rangle = \sum_{\tilde{n}}\partial_k c_{\tilde{n}}(k)\langle\{O_{\tilde{n}}\}_k\rangle = \langle\partial_k\sum_{\tilde{n}} c_{\tilde{n}}(k)\{O_{\tilde{n}}\}_k\rangle. \tag{4.7}$$

The connection can in principle be found in any other basis by simple computation. As far as the basis $O_{\tilde{n}}(\varphi_k)$ is concerned it is simpler to check directly that (4.4) satisfies (4.5). For this end we insert the arbitrary operator

$$O_k = \sum_{\tilde{m}} c_{\tilde{m}}(k)O_{\tilde{m}}(\varphi_k) = \underline{c}(k)\underline{O} \tag{4.8}$$

into (4.5),

$$\partial_k \left(\underline{c}(k)\langle\underline{O}\rangle\right) = \langle D_k\underline{c}(k)\underline{O}\rangle, \tag{4.9}$$

and write, with Eq. (4.4),

$$\partial_k\underline{c}(k)\langle\underline{O}\rangle + \underline{c}(k)\partial_k\langle\underline{O}\rangle = \partial_k\underline{c}(k)\langle\underline{O}\rangle - \langle\underline{O}\rangle_k\underline{\underline{\Gamma}}\underline{c}(k). \tag{4.10}$$

In order to prove (4.5), we need the relation

$$-\partial_k\langle\underline{O}\rangle = \langle\underline{O}\rangle\underline{\underline{\Gamma}}. \tag{4.11}$$

Using $S_{k=0}(G_{\tilde{n}}(k))$ as the generator function for $\langle O_{\tilde{n}}\rangle$, the l.h.s. of Eq. (4.11) can be written as

$$\begin{aligned}\frac{1}{\delta k}\left[\frac{\delta S_0}{\delta G_{\tilde{m}}(k-\delta k)} - \frac{\delta S_0}{\delta G_{\tilde{m}}(k)}\right] &= \frac{1}{\delta k}\left[\frac{\delta S_0}{\delta G_{\tilde{m}}(k-\delta k)} - \sum_{\tilde{n}}\frac{\delta S_0}{\delta G_{\tilde{n}}(k-\delta k)}\frac{\delta G_{\tilde{n}}(k-\delta k)}{\delta G_{\tilde{m}}(k)}\right]\\ &= \frac{1}{\delta k}\sum_{\tilde{n}}\frac{\delta S_0}{\delta G_{\tilde{n}}(k-\delta k)}\left[\delta_{\tilde{n}\tilde{m}} - \frac{\delta G_{\tilde{n}}(k-\delta k)}{\delta G_{\tilde{m}}(k)}\right] = \frac{1}{k}\sum_{\tilde{n}}\frac{\delta S_0}{\delta G_{\tilde{n}}(k)}\frac{\delta\beta_{\tilde{n}}(k)}{\delta G_{\tilde{m}}(k)},\end{aligned} \tag{4.12}$$

in the limit $\delta k \to 0$. According to (3.19), the last line agrees with the r.h.s. of Eq. (4.11).

Let us turn now to the usual quantum field theoric problem without inhomogeneous external sources, that have only been introduced as technical tools, i.e. take the limit $g_{\tilde{n}}(k) \to 0$. Then the operator mixing matrix and the connection reduce to

$$S_{\tilde{m}\tilde{n}}(k,\Lambda) = \delta(x_m - x_n)s_{mn}(k,\Lambda), \qquad \Gamma_{\tilde{m}\tilde{n}}(k) = \delta(x_m - x_n)\gamma_{mn}(k), \tag{4.13}$$

resp. with the coordinate independent matrices

$$s_{mn}(k,\Lambda) = \frac{\partial g_m(k)}{\partial g_n(\Lambda)}, \qquad k\gamma_{mn}(k) = \frac{\partial\beta_m(k)}{\partial g_n(k)} \tag{4.14}$$

and the RG equations for the operator mixing matrix take the form

$$\partial_k s_{mn}(k,\Lambda) = \sum_{\ell}\gamma_{m\ell}(k)s_{\ell n}(k,\Lambda). \tag{4.15}$$

V. RG FLOW AND UNIVERSALITY

It has been found that the operator mixing, the problem of keeping the expectation values cut-off independent can be handled by a linear transformation, the parallel transport of the operators. We shall show in this section that the salient features of the RG flow can be recovered from this parallel transport alone.

The scaling combinations of the coupling constants are introduced traditionally in the vicinity of a fixed point $G^*_{\tilde{m}}$ in the space of the coupling constants which have been made dimensionless by the help of the cut-off k. The basic assumption of the RG strategy is that the evolution equations can be linearized around the fixed points,

$$\beta_{\tilde{n}} \approx \sum_{\tilde{m}}\Gamma^*_{\tilde{n}\tilde{m}}(G_{\tilde{m}} - G^*_{\tilde{m}}). \tag{5.1}$$

The relevant, marginal and irrelevant coupling constants are the superpositions

$$G^{sc}_{\tilde{n}} = \sum_{\tilde{m}}\bar{c}_{\tilde{n}\tilde{m}}\left(G_{\tilde{m}} - G^*_{\tilde{m}}\right) \tag{5.2}$$

made by the left eigenvectors of Γ,

$$\sum_{\tilde{m}}\bar{c}_{\tilde{n}\tilde{m}}\Gamma^*_{\tilde{m}\tilde{r}} = \alpha_{\tilde{n}}\bar{c}_{\tilde{n}\tilde{r}}, \tag{5.3}$$

with $\alpha_{\tilde{n}} < 0$, $\alpha_{\tilde{n}} = 0$ and $\alpha_{\tilde{n}} > 0$, resp. The scale dependence of the scaling coupling constants $G^{sc}_{\tilde{n}}$ is

$$G^{sc}_{\tilde n} \sim k^{\alpha_{\tilde n}}. \tag{5.4}$$

Let us consider a local operator of the form

$$O(\phi(x)) = \sum_n b_n O_n(\phi(x)) \tag{5.5}$$

where $O_n(\phi(x))$ is the product of the terms $\partial_{\mu_1}\cdots\partial_{\mu_\ell}\phi^m(x)$. It will be necessary to separate for the possible scale dependent rescaling factors. For this end we introduce the norm

$$||O|| = \sqrt{\sum_n b^2_n} \tag{5.6}$$

and adopt the convention that the coupling constants $G_{\tilde n}(\Lambda)$ of the bare action always multiply local operators with unit norm, $||O_{\tilde n}(\Lambda)|| = 1$. As long as we consider local operators the norm defined above is sufficient and no index with continuous range is needed in its definition. It will be useful to consider the normalized operator flow,

$$\overline{O} = \frac{O}{||O||}, \tag{5.7}$$

in addition to the original one, O.

The scaling operators

$$O^{sc}_{\tilde n} = \sum_{\tilde m} c_{\tilde n\tilde m} O_{\tilde m} \tag{5.8}$$

are obtained by means of the right eigenvectors of Γ^*,

$$\sum_{\tilde m} \Gamma^*_{\tilde r\tilde m} c_{\tilde m\tilde n} = \alpha_{\tilde r} c_{\tilde r\tilde n}, \tag{5.9}$$

satisfying the conditions of completeness $c\cdot\bar c = 1$ and orthonormality $\bar c\cdot c = 1$. The coupling constants of the action

$$S_k = \sum_{\tilde n} G_{\tilde n}(k)\overline{O^{sc}_{\tilde n}(\phi_k)} \tag{5.10}$$

obviously follow (5.4). The operator

$$O = \sum_{\tilde n} b_{\tilde n}\overline{\{O^{sc}_{\tilde n}\}_k}, \tag{5.11}$$

written at scale k in this basis yields the parallel transport trajectory

$$\{O\}_{k'} = \sum_{\tilde n} b_{\tilde n}\{\overline{\{O^{sc}_{\tilde n}\}_k}\}_{k'} = \sum_{\tilde n} b_{\tilde n}\left(\frac{k'}{k}\right)^{\alpha_{\tilde n}}\overline{\{O^{sc}_{\tilde n}\}_k} \tag{5.12}$$

in the vicinity of the fixed point.

The question we explore now is what information does the flow $\{O\}_k$ contain about the importance of certain interactions as the function of the observational scale k. Let us start with the remark that according to (3.13) the fixed point of the blocking relation, where the action, expressed in terms of the dimensionless coupling constants, is scale independent agrees with the fixed point of the operator blocking. The linearization around the fixed points renders the critical exponents of the coupling constants and the scaling operators equivalent since the left and the right spectrum of Γ^* agree. The operators whose structure converges and changes by an overall factor only are the scaling operators. The corresponding critical exponents can be read off from the evolution of their norm.

The concept of universality stands for the independence of the dimensionless quantities of the long distance physics from the initial value of the irrelevant coupling constants in the UV scaling regime. The operator $\{O_{\tilde n}(\varphi_k)\}_k$ constructed from the local terms of the bare action by parallel transport represents the influence of the bare coupling constants on the physics of the scale k. Since the action is dimensionless only relevant operators have non-vanishing parallel transport flow at finite scale according to Eq. (3.13), the parallel transport of the irrelevant operators which are made dimensionless by the running scale k vanishes. Naturally this does not mean that the effective theories are renormalizable since a renormalizable (relevant) operator parallel transported down from the cut-off mixes with non-renormalizable (irrelevant) ones, as well.

The operator flow is particularly well suited for the studies of models whose RG flow visits different scaling regimes. In general, any renormalizable theory without manifest scale invariance possesses at least two scaling regimes, one around the UV and another one at the IR fixed points which are separated by a crossover at the intrinsic scale of the theory, $k = k_{cr}$. For models with mass gap the IR scaling regime is trivial, i.e. the relevant operators are Gaussian. Imagine a model with dynamically generated intrinsic scale, e.g. with spontaneous symmetry breaking or condensation or dimensional transmutation, where the IR instability generates non-trivial scaling laws and non-Gaussian relevant coupling constants appear in the IR scaling. Let us suppose that there is a non-renormalizable operator O which becomes relevant in the IR regime and consider its parallel transport globally, from the UV to the IR fixed point [8]. Since O is non-renormalizable $||\{O\}_k||$ decreases as k is lowered in the UV reflecting the diminishing importance of a non-renormalizable operator well below the cut-off. But after having crossed the crossover the flow reflects the properties of a relevant operator, i.e. $||\{O\}_k||$ increases as k is further lowered in the IR scaling regime. Now it becomes a competition between the UV suppression and the IR enhancement to form the final sensitivity on the cut-off scale parameter. Since the IR increase of the norm is usually fed by IR or collinear divergences the length of the scaling regimes are determined by Λ/k_{cr} and Lk_{cr} where L is the size of the system, an IR cut-off. If the coherence is not lost for sufficiently large distances then the condition $||\{O\}_k|| = 1$ can be reached at low enough k for any value of the UV cut-off, i.e. the IR instability can make the otherwise weak sensitivity on the short distance physics strong.

Finally, few words are in order about the possibility of determining the sensitivity matrix. There is a *direct* method by solving the set (3.19). Namely, the right hand sides of Eq. (3.10) are analytic expressions for the beta-functions that can be differentiated analytically, and the numerical values of the connection matrix then easily computed and used as input for Eq. (3.19). There is also an *indirect* method according to the definition (3.15). One has to solve the Wegner-Houghton equations (3.10) for the blocked sources as functionals of their initial values at the scale Λ, and differentiate the solution w.r.t. these initial conditions. This latter method was used in Ref. [8] to show that the ϕ^4 model in the phase with spontaneously broken symmetry does in fact possess a non-renormalizable relevant operator in the IR scaling regime.

VI. RG AND RENORMALIZED PERTURBATION EXPANSION

The comparison of the renormalisation of the composite operators presented above with the usual operator mixing obtained in the framework of the renormalized perturbation expansion serves two goals. First, it shows that the usual operator mixing represents the evolution of the composite operators towards the UV direction. Second, it helps to understand an apparent paradox, namely that non-renormalizable operators can be "renormalized" within a renormalizable theory. The composite operator renormalization stands for the program of finding the counterterms which renders the Green's functions even with the given composite operator insertions finite as the cut-off is removed. Since the finiteness of the Green's functions implies the finiteness of the partition functions where the composite operators are introduced with a source term in the action, as in Eq. (3.9), the completion of this program would amount to the renormalization of theories where the composite operators appear in the action.

A. Renormalized Perturbation Expansion

Let us start with the bare action (3.1),

$$S[\phi, G_{\tilde{n}}] = \sum_{\tilde{n}} G_{\tilde{n}} O_{\tilde{n}}(\phi). \tag{6.1}$$

For the sake of simplicity, we neglected the subscript Λ here, but the dependence on the external sources $G_{\tilde{n}}$ is made explicit; $O_{\tilde{n}}(\phi)$ is a complete set of normalized bare operators. The UV and IR momentum cut-offs, Λ and k are assumed to be introduced for the field variable $\phi(x)$ in order to achieve a better comparison with the RG method. Let us separate the zero modes of the sources, i.e. the bare coupling constants g_n, $G_{\tilde{n}} = g_n + g_{\tilde{n}}$. Furthermore, specify $O_{(1|1)}(\phi(x)) = -\frac{1}{2}\phi(x)\Box\phi(x)$, $O_1(\phi(x)) = \phi(x)$, $g_{(1|1)} = Z_\phi$, $g_1 = 0$. Then, $G_{\tilde{1}} = g_{\tilde{1}} = -j(x)$ is the external current coupled to the bare field.

The corresponding quantum field theory is defined by the generating functional

$$Z[G_{\tilde{n}}] = \frac{\int \mathcal{D}\phi e^{-S[\phi, G_{\tilde{n}}]}}{\int \mathcal{D}\phi e^{-S[\phi; g_n]}}. \tag{6.2}$$

Then, we obtain the generating functionals $W[G_{\tilde{n}}] = \ln Z[G_{\tilde{n}}]$ and $\Gamma[\varphi,\, G_{\tilde{n},n\neq 1}] = -W[G_{\tilde{n}}] + \int dx j(x)\varphi(x)$ of the connected and 1PI Green's functions, resp., with $\varphi(x) = \delta W/\delta j(x)$.

The insertion of the operator $O_{\tilde{n}}(\phi)$ in the 1PI Green's functions is obtained via functional derivation w.r.t. the corresponding source $G_{\tilde{n}}$, and the identities

$$\langle O_{\tilde{n}}(\phi)\rangle_{1PI} = -\frac{1}{Z}\frac{\delta Z}{\delta G_{\tilde{n}}}\bigg|_0 = -\frac{\delta W}{\delta G_{\tilde{n}}}\bigg|_0 = \frac{\delta \Gamma}{\delta G_{\tilde{n}}}\bigg|_0 \tag{6.3}$$

hold for $n \neq 1$. The subscript $\ldots|_0$ stands for $g_{\tilde{n}} \equiv 0$ (and $\varphi = 0$ for Γ). Owing to its definition via the bare action, this vacuum expectation value is expressed in terms of the bare couplings.

The theory defined above is non-renormalizable. One has to satisfy infinitely many renormalization conditions in order to fix the renormalized values g_{nR} of the infinitely many coupling constants. This is equivalent to specifying a particular RG trajectory in the RG approach. The theory with non-renormalizable coupling constants does not allow the removal of the cut-off, i.e. the extrapolation towards short distances is problematic. But as long as the properties far away from the cut-off towards the IR direction are concerned and the possible non-trivial effects of the IR scaling regime [8] are neglected, the values of the non-renormalizable coupling constants at the cut-off effect the overall scale of the theory only. When considering dimensionless quantities this scale drops out and the non-renormalizable coupling constants can be set at the cut-off in an arbitrary manner. As a result, there is no problem to construct the operator mixing below a sufficiently high cut-off. One should bear in mind that even the renormalizable theories contain non-renormalizable operators, the regulator. In fact, the comparison of a theory with different regularizations shows that the regulators amount to a set of irrelevant operators when written in the action. These regulator terms have tree-level fine-tuning which, according to the universality, is sufficient to keep the cut-off independent dynamics fixed. In what follows we take the usual point of view and the regulators will not be represented in the action.

The renormalization conditions enable one to rewrite the bare action as the sum of the renormalized terms and that of infinitely many counterterms

$$S[\phi, G_{\tilde{n}}] = \sum_{\tilde{n}} \left(G_{\tilde{n}R} + c_{\tilde{n}}[G_{\tilde{m}R}]\right) O_{\tilde{n}}(\phi) \equiv S_R[\phi, G_{\tilde{n}R}] \tag{6.4}$$

in the framework of the RPE. Here we introduced the renormalized sources via $G_{\tilde{n}} = G_{\tilde{n}R} + c_{\tilde{n}}[G_{\tilde{m}R}]$. As mentioned above, all but finite counterterms influence an overall scale factor only of the theory according to the universality. The cut-off will be kept arbitrary large but finite and fixed. Below the notation $c_{(1|1)}[G_{\tilde{m}R}] = Z_\phi - 1$ and the renormalization condition $G_{(1|1)R} \equiv 1$ are used. The coefficients of the counterterms $c_{\tilde{n}}$ are functionals of the renormalized sources $G_{\tilde{m}R}$. Without loss of generality we can assume that all external sources are cut off at some momentum $\Lambda_s \ll \Lambda$. Then, the counterterms remain local similarly to the case with constant external sources [6] and the coefficients $c_{\tilde{n}}$ are only affected by the zero modes of the sources, i.e. they are independent of the spacetime coordinate x and are functions of the coupling constants g_{nR} and the UV and IR cut-offs, Λ and k, resp. Then, the relations $g_{nR} + c_n(g_{mR}) = g_n$ and $g_{\tilde{n}R} \equiv g_{\tilde{n}}$ hold. For $n = 1$ these yield $-g_{\tilde{1}} \equiv j(x) = j_R(x)$ and $g_1 = c_1$ for the renormalization condition $g_{1R} = 0$. We see that $g_{\tilde{n}} \equiv 0$ implies $g_{\tilde{n}R} \equiv 0$ and vice versa.

The renormalized generating functional Z_R considered as the functional of the renormalized sources,

$$Z_R[G_{\tilde{n}R}] = \frac{\int \mathcal{D}\phi(x) e^{-S_R[\phi, G_{\tilde{n}R}]}}{\int \mathcal{D}\phi(x) e^{-S_R[\phi; g_{nR}]}} = Z[G_{\tilde{n}}] \tag{6.5}$$

is the generating functional of the Green's functions with renormalized composite operator insertions. The following equations hold:

$$W_R[G_{\tilde{n}R}] = \ln Z_R[G_{\tilde{n}R}] = W[G_{\tilde{n}}], \tag{6.6}$$

and

$$\Gamma_R[\varphi, G_{\tilde{n}\neq\tilde{1},R}] = -W_R[G_{\tilde{n}R}] + \int dx j_R(x)\varphi(x) = -W[G_{\tilde{n}}] + \int dx j(x)\varphi(x) = \Gamma[\varphi, G_{\tilde{n}\neq\tilde{1}}], \tag{6.7}$$

where $\varphi(x) = \delta W_R/\delta j_R(x) = \delta W/\delta j(x)$.

The renormalized composite operator insertion $[O_{\tilde{n}}(\phi)]_R$ ($n \neq 1$) is obtained by functional derivation w.r.t. the renormalized external source $G_{\tilde{n}R}$:

$$\langle [O_{\tilde{n}}(\phi)]_R\rangle_{1PI} = -\frac{1}{Z_R}\frac{\delta Z_R[G_{\tilde{m}R}]}{\delta G_{\tilde{n}R}}\bigg|_0 = -\frac{\delta W_R[G_{\tilde{m}R}]}{\delta G_{\tilde{n}R}}\bigg|_0 = \frac{\delta \Gamma_R[\varphi, G_{\tilde{m}\neq\tilde{1},R}]}{\delta G_{\tilde{n}R}}\bigg|_0. \tag{6.8}$$

Now we find the following relations

$$\left.\frac{\delta\Gamma_R[\varphi, G_{\tilde{m}\neq\tilde{1},R}]}{\delta G_{\tilde{n}R}}\right|_0 = \left.\frac{\delta\Gamma[\varphi, G_{\tilde{m}\neq\tilde{1}}]}{\delta G_{\tilde{n}R}}\right|_0 = \sum_{\tilde{r}\neq\tilde{1}} \left.\frac{\delta G_{\tilde{r}}}{\delta G_{\tilde{n}R}}\right|_0 \left.\frac{\delta\Gamma[\varphi, G_{\tilde{m}\neq\tilde{1}}]}{\delta G_{\tilde{r}}}\right|_0 . \tag{6.9}$$

Making use of the derivative

$$\left.\frac{\delta G_{\tilde{m}}}{\delta G_{\tilde{n}R}}\right|_0 = (Z^{-1})_{mn}\delta(x-y) \tag{6.10}$$

with the operator mixing matrix

$$(Z^{-1})_{mn} = \delta_{mn} + \frac{\partial c_m(g_{rR})}{\partial g_{nR}} \tag{6.11}$$

(for $n, m \neq 1$), one finds

$$[O_{\tilde{n}}(\phi)]_R = \sum_{m\neq 1} O_{\tilde{m}}(\phi)\left(Z^{-1}\right)_{mn} \tag{6.12}$$

for $n \neq 1$. This can be extended to $n, m = 1$ by defining $(Z^{-1})_{1m} = \delta_{m1}$, $(Z^{-1})_{n1} = \delta_{1n}$. It is worthwhile mentioning that the operator mixing matrix turned out local in spacetime coordinates, i.e. it is independent of the momentum at which the composite operator insertion is taken. The matrix Z_{nm} is just the transposed of that used in Ref. [6].

B. Comparison of the two schemes

We compare now the notion of renormalized operator in the perturbative approach with the notion of blocked operator in the RG framework. Table I summarizes the formal similarities between the two approaches.

The RG approach keeps the UV cut-off Λ fixed and uses a decreasing IR cut-off k, so that the RG trajectories are passed towards the IR limit $k \to 0$. On the contrary, the couplings are defined at some low-energy scale $k = \mu \ll \Lambda$ and the UV cut-off is shifted towards infinity in the RPE, and the RG trajectories are followed just in the opposite direction, towards large momenta. The RG approach reproduces the perturbative results for the ordinary Green's functions in the UV scaling regime [6,7] up to powers and logarithms of μ/Λ which are vanishing in the asymptotic limit $\Lambda \to \infty$.

It is easy to see that

$$\sum_{\tilde{n}} G_{\tilde{n}}(\Lambda)\{O_{\tilde{n}}\}_k = \sum_{\tilde{n}} G_{\tilde{n}}(\Lambda)\frac{\delta S_k[\phi_k]}{\delta G_{\tilde{n}}(k)} \tag{6.13}$$

gives the blocked action with cut-off k in the order $\mathcal{O}(G(\Lambda))$, the blocking of the action and the operators agree in the linearized level, i.e. they share the fixed points and the critical exponents. The Legendre transformed effective action with the IR cut-off $k \neq 0$ [9,12] describes the effective theory after the high-frequency modes have been eliminated, similarly to the blocked action for the low-frequency modes.

The RPE deals with the effective action in the limit $k \to 0$, $\Lambda \to \infty$. Since the bare couplings g_n and the renormalized couplings g_{nR} are the analogues of $g_n(\Lambda)$ at the UV cut-off scale and of the blocked couplings $g_n(k)$, resp., the operator mixing matrix $s_{nm} = \partial g_n(k)/\partial g_m(\Lambda)$ defined in the RG approach is just the analogue of the matrix Z_{nm} defined via $(Z^{-1})_{mn} = \partial g_m/\partial g_{nR}$ in the RPE. This analogy will be demonstrated in Sect. VII on the equivalence of the one-loop perturbative results with those obtained by the RG method in IMA for a few elements of the operator mixing matrix in the particular case of ϕ^3 theory in dimension $d = 6$.

The analogues of the blocked operators $\{O_{\tilde{n}}(\varphi_k)\}_k$ are not used in the RPE. In the latter one seeks the operator at the scale of the UV cut-off that reproduces the expectation value of an operator given at the renormalization scale. The renormalized operator $[O_{\tilde{n}}(\phi)]_R$ satisfying this requirement is the analogue of the operator $[O_{\tilde{n}}]_k(\phi)$ introduced in the RG approach by means of inverse blocking.

It is the basic advantage of the RG approach that a non-perturbative answer can be obtained with its help on operator mixing, whereas also the UV finite pieces of the operator mixing matrix are automatically determined. Furthermore, the RG approach enables one to get a deeper insight in the reason of operator mixing as a natural consequence of reproducing the same vacuum expectation values in the bare theory and in the blocked one, or in other words as a direct consequence of integrating out degrees of freedom.

VII. OPERATOR MIXING IN ϕ_6^3 THEORY

The relation between the composite operator renormalization in the RG scheme and the operator mixing of the RPE is demonstrated in this section in the case of a simple scalar model.

A. RG method

We start with the determination of the blocked operators in the framework of the ϕ^3 theory in dimension $d = 6$ in the independent mode approximation (IMA) of the next-to-leading order of the derivative expansion and show that both the direct and the indirect methods lead to the same results.

Let us include as base operators $O_{\tilde{n}}(\varphi_k)$ the derivative operators

$$D_{(0|1)}(\varphi_k) = -\Box\varphi_k, \quad D_{(1|1)}(\varphi_k) = -\frac{1}{2}\varphi_k\Box\varphi_k, \quad D_{(0|2)}(\varphi_k) = -\frac{1}{2}\Box\varphi_k^2 \tag{7.1}$$

and the local potential

$$V(\varphi_k) = \sum_{\ell=0}^{\infty} g_\ell(k)\frac{\varphi_k^\ell}{\ell!} \tag{7.2}$$

into the Ansatz for the blocked action:

$$S_k = \int d^dx \left[Z_{(0|1)}(k)D_{(0|1)}(\varphi_k) + Z_{(1|1)}(k)D_{(1|1)}(\varphi_k) + Z_{(0|2)}(k)D_{(0|2)}(\varphi_k) + V(\varphi_k)\right]. \tag{7.3}$$

The bare ϕ^3 theory is specified by the bare couplings $g_2(\Lambda) = m^2$, $g_3(\Lambda) = \lambda$, $g_{\ell\geq 4}(\Lambda) = 0$, $Z_{(0|1)}(\Lambda) = 0$, $Z_{(1|1)}(\Lambda) = 1$, and $Z_{(0|2)}(\Lambda) = -\frac{1}{2}$.

The choice of operators in (7.3) means that the field dependence of the wave function renormalization is not taken into account. Terms with higher order derivatives of the field are also neglected, but the extension of the operator basis is straightforward. Other operators of the discussed types can be expressed as linear combinations of those included in the basis, e.g.

$$-\partial_\mu\varphi_k \cdot \partial_\mu\varphi_k = D_{(0|2)}(\varphi_k) - 2D_{(1|1)}(\varphi_k). \tag{7.4}$$

Having derived the explicit forms of the right hand sides of Eqs. (3.8) and (3.19), the limit $g_{\tilde{n}}(k) \to 0$ can already be taken at the beginning of the calculation except for the couplings multiplying the operators $\Box\varphi_k(x)$ and $\Box\varphi_k^2(x)$. The corresponding terms in the action would yield pure surface terms and therefore, their effects on operator mixing can only be kept track if the corresponding inhomogeneous sources are replaced by constants only at the end of the calculation.

A further remark is in order here. Namely, we have formulated the whole procedure in the Wegner-Houghton framework with a sharp cut-off and used derivative expansion. It is, however, well-known that the sharp cut-off introduces undesirable singularities due to the derivatives of the step like cut-off function [11]. This may also cause our method in its present form with sharp cut-off to fail in correctly describing the renormalization of the derivative operators. We do not see, however, any objections to reformulate our method of treating composite operator renormalization using a smooth cut-off on the base of Polchinski's equation [7].

1. *Indirect method*

For the indirect determination of the operator mixing matrix the following steps should be performed:

1. **Determination of the blocked couplings.** Our method to establish the coupled set of differential equations (3.10) for the couplings from Eq. (3.8) is similar to that used in [10]. Namely, we substitute $\varphi_k(x) = \varphi_0 + \eta(x)$ where φ_0 is an arbitrary constant, expand both sides of Eq. (3.8) in Taylor series w.r.t. the inhomogeneous piece $\eta(x)$, and compare the coefficients of the corresponding operators $O_{\tilde{n}}(\eta)$. Since only operators with second derivatives are considered, it is sufficient to terminate the expansion at the quadratic terms. In Appendix A, the set of coupled first order differential equations (A.11), (A.15), (A.16), and (A.24) is obtained for the blocked couplings. These equations are then solved in independent mode approximation analytically in Appendix B.

2. **Determination of the operator mixing matrix in IMA** through differentiating the solutions for the blocked couplings w.r.t. the initial values of the various couplings at the scale Λ. This step is discussed in detail in App. B. The orders of magnitude of the elements of the operator mixing matrix w.r.t. the UV cut-off Λ are indicated in Table II.

2. Direct method

As byproducts, the right hand sides of Eqs. (A.11), (A.15), (A.16), and (A.24) provide us the exact analytic expressions for the beta-functions, $\beta_n(k)$ (see the beginning of Appendix C). Thus, analytic expressions can be obtained for the elements of the connection γ_{nm} by differentiating the appropriate beta-functions w.r.t. the appropriate coupling constants without any approximation. The results are summarized in Appendix C and the non-trivial matrix elements indicated in Table III. This matrix is the input for Eqs. (3.19) to the direct determination of the operator mixing matrix.

Eqs. (3.19) can be rewritten as integral equations in a rather compact form introducing the coloumn vectors $\underline{s}_n$ with the elements s_{mn} (with the row index m) and the connection matrix $\underline{\underline{\gamma}}$ (with the elements γ_{lm}):

$$\underline{s}_n(k) = \underline{s}_n(\Lambda) - \int_k^\Lambda d\kappa \underline{\underline{\gamma}}(\kappa)\underline{s}_n(\kappa). \tag{7.5}$$

This means that all the coloumn vectors $\underline{s}_n(k)$ satisfy the same ordinary first order linear differential equation, only the initial conditions $\underline{s}_n(\Lambda)$ are different for them. The analytic expressions for the elements of the connection matrix (see Appendix C) should be used as input.

Here we determine the operator mixing matrix analytically in IMA. For this approximation, all the coupling constants in the explicit expressions for the elements of $\underline{\underline{\gamma}}$ and the coloumn vector $\underline{s}_n(\kappa)$ should be replaced by their bare values on the r.h.s. of Eq. (7.5). For the $\lambda\phi^3$ theory one has $Z_{(1|1)}(\Lambda) = 1$, $Z_{(0|1)}(\Lambda) = 0$, $Z_{(0|2)}(\Lambda) = -\frac{1}{2}$, $g_2(\Lambda) = m^2$, $g_3(\Lambda) = \lambda$, $g_{\ell\geq 4}(\Lambda) = 0$. Since there is no operator mixing at the scale Λ, the initial conditions are $s_{mn}(\Lambda) = \delta_{mn}$. Then, Eqs. (7.5) take the forms

$$s_{mn}^{IMA}(k) = \delta_{mn} - \int_k^\Lambda d\kappa \gamma_{mn}^{IMA}(\kappa), \tag{7.6}$$

and their solutions can be expressed in terms of the integrals in App. E. Using the results of App. C, it has been checked that the solutions are just the operator mixing coefficients found in App. B by the indirect method previously.

B. Perturbative approach

In App. D we determine perturbatively the operator mixing coefficients $(Z^{-1})_{nm}$ for ϕ^3 theory in dimension $d = 6$ in one-loop approximation. The inverse of $(Z^{-1})_{nm}$ can directly be compared with the results obtained for the matrix s_{mn} by means of the RG approach in IMA in Sect. VII A. The inversion of the matrix results in the change of the sign of the terms of $o(\hbar)$ of the matrix elements. Expressing the perturbative results in terms of the loop integrals given in App. E, it is easy to recognize that $Z_{nm} = s_{nm}(k = 0)$ in the above mentioned approximations. This agreement illustrates how the perturbative approach is related to the RG approach, that was discussed in Sect. VI B.

VIII. SUMMARY

In the toy model of a zero-dimensional field theory the operator mixing has been explained as the natural consequence of integrating out degrees of freedom. Then, the notion of blocked operators is defined in one-component scalar field theory through the requirement of reproducing the same vacuum expectation value in the bare theory and in the blocked (effective) one. The blocking procedure proposed by Wegner and Houghton has been used, i.e. the high-frequency degrees of freedom were integrated out in infinitesimal momentum shells sequentially using a sharp moving cut-off.

It is shown that the blocking of operators introduced in this paper satisfies the (semi)group property. Differential equations are derived for the elements of the operator mixing matrix that should be solved simultaneously with the Wegner-Houghton equations. It is found that the blocking of operators corresponds to parallel transporting them along the RG trajectory, a flat, one-dimensional manifold. The scale dependence of the operator mixing matrix is governed by the connection, showing the changes of the beta-functions due to infinitesimal changes of the couplings at a certain scale k. It is also shown that the eigenoperators of the connection are the local scaling operators in any scaling regimes. Thus, solving the eigenvalue problem of the connection enables one in principle to detect different scaling regimes and find the corresponding relevant operators.

The limitations are, of course, the validity region of the Wegner-Houghton equation due to occurring a non-trivial saddle point, and the usage of the sharp cut-off together with the gradient expansion. As to the latter technical problem, a generalization of operator blocking to a smooth cut-off approach seems to be possible by including the appropriate cut-off terms into the action. The reformulation of the whole issue in the framework of renormalization in the internal space [12] would solve both of the above mentioned problems and do also for generalization to gauge theories.

The differential RG approach and the perturbative approach for operator renormalization are compared in detail. It is explained that the renormalized operator (used in perturbative terms) corresponds to choosing an operator at the renormalization scale and looking for an operator at the UV scale (tending to infinity) that reproduces the vacuum expectation value of the chosen operator. On the particular example of ϕ^3 theory in dimension $d = 6$ the agreement of the results of the RG approach in IMA with the one-loop perturbative ones has been illustrated for the elements of the operator mixing matrix in a truncated basis of operators.

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APPENDIX A: RG EQUATIONS FOR THE BLOCKED COUPLINGS

The RG equations for the blocked couplings figuring in the blocked action (7.3) are derived by separating the zero mode φ_0 of the field $\varphi_k(x)$, $\varphi_k(x) = \varphi_0 + \eta(x)$ and expanding both sides of Eq. (3.8) in Taylor series w.r.t. $\eta(x)$ terminated at the quadratic terms. The zeroth, first, and second order terms are

$$\sigma_0 = \int d^d x V(\varphi_0),$$

$$\sigma_1 = \int d^d x \left\{ - \left[Z_{(0|1)}(x,k) + \varphi_0 Z_{(0|2)}(x,k) \right] \Box\eta + V^{(1)}(\varphi_0)\eta(x) \right\}, \tag{A.1}$$
$$\sigma_2 = \int d^d x \left[-Z_{(1|1)}(k)\frac{1}{2}\eta\Box\eta - Z_{(0|2)}(x,k)\frac{1}{2}\Box\eta^2 + \frac{1}{2}V^{(2)}(\varphi_0)\eta^2 \right].$$

Furthermore, we need the second derivative of the blocked action, $S_k^{(2)} = A + B + C + o(\eta^3)$ with the matrices A, B, and C of zeroth, first, and second order of η and having the following forms:

$$A_{pq} = \mathcal{A}(p^2)V_d\delta_{p+q} + \chi_{pq} \tag{A.2}$$

with the diagonal part

$$\mathcal{A}(p^2) = V_\kappa^{(2)}(\varphi_0) + Z_{(1|1)}p^2, \tag{A.3}$$

and the off-diagonal piece

$$\chi_{pq} = \int d^d z Z_{(0|2)}(z)(p+q)^2 e^{i(p+q)z}, \tag{A.4}$$

furthermore

$$B_{pq} = \mathcal{B}\int dz\eta(z)e^{i(p+q)z} \tag{A.5}$$

with $\mathcal{B} = V^{(3)}(\varphi_0)$, and

$$C_{pq} = \frac{1}{2}V^{(4)}(\varphi_0)\int d^d z\eta^2 e^{i(p+q)z}. \tag{A.6}$$

Rewriting the logarithm on the r.h.s. of Eq. (3.8) as $\ln(A+B+C+\ldots) = \ln A + \ln\left[1 + A^{-1}(B+C+\ldots)\right]$ and expanding the second logarithmic term in Taylor series, $A^{-1}B + A^{-1}C - \frac{1}{2}A^{-1}BA^{-1}B\ldots$, one finds the equations

$$k\partial_k\sigma_0 = -k^d\alpha_d\int\frac{d\omega_n}{\Omega_d}\left(\ln A\right)_{kn,-kn}, \tag{A.7}$$
$$k\partial_k\sigma_1 = -k^d\alpha_d\int\frac{d\omega_n}{\Omega_d}\left(A^{-1}B\right)_{kn,-kn}, \tag{A.8}$$
$$k\partial_k\sigma_2 = -k^d\alpha_d\int\frac{d\omega_n}{\Omega_d}\left(A^{-1}C - \frac{1}{2}A^{-1}BA^{-1}B\right)_{kn,-kn}. \tag{A.9}$$

Here, the matrix products of the form $(MN)_{pq} = \sum_P M_{p,-P}N_{P,q}$ should be understood over a restricted phase space, i.e., the sum over P should be restricted to the thin momentum shell $k < |P| \le k + \delta k$. Such sums are performed in the continuum limit as integrals over the interval $[k-\epsilon, k+\delta k]$ with $\epsilon > 0$ infinitesimal and the limits $\epsilon \to 0$ and $\delta k \to 0$ are taken at the end. It has been checked that this procedure provides a result for field independent wave function renormalization $Z_{(1|1)}$ in IMA which is in agreement with the perturbative one-loop result obtained e.g. in [6] for ϕ^3 theory in $d = 6$ dimension.

$\underline{\sigma_0}$: In order to find the explicit form of Eq. (A.7), we write

$$(\ln A)_{p,-p} = \text{const.} + \ln\mathcal{A}(p^2) + \frac{1}{V_d}(\mathcal{A}^{-1}\chi)_{p,-p} - \frac{1}{2V_d^2}(\mathcal{A}^{-1}\chi\mathcal{A}^{-1}\chi)_{p,-p} + o(\chi^3) \tag{A.10}$$

Since χ contains second powers of the momenta, its first, second, etc. powers generate gradient terms of the order ∂^2, ∂^4, etc., resp. In the second order of the gradient expansion we only have to include the term linear in χ, but its diagonal matrix element vanishes, so that we obtain $(\ln A)_{p,-p} \approx \ln\mathcal{A}(p^2)$ and the integro-differential equation

$$V_k(\varphi_0) = V_\Lambda(\varphi_0) + \hbar\alpha_d\int_k^\Lambda d\kappa\,\kappa^{d-1}\ln\mathcal{A}(\kappa^2) \tag{A.11}$$

for the blocked potential $V_k(\varphi_0)$. Here $\alpha_d = \Omega_d(2\pi)^{-d}/2$, $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the entire solid angle in the d dimensional momentum space.

$\underline{\sigma_1:}$ For the evaluation of the r.h.s. of Eq. (A.8) we need the inverse of the non-diagonal matrix A_{pq}. Expanding it in powers of the off-diagonal matrix χ_{pq}, one finds

$$(A^{-1})_{pq} = \frac{1}{V_d}\mathcal{A}^{-1}(p^2)\delta_{p+q} - \frac{1}{V_d^2}\mathcal{A}^{-1}(p^2)\chi_{pq}\mathcal{A}^{-1}(q^2) + o(\chi^2). \tag{A.12}$$

The trace of the matrix product

$$(A^{-1}B)_{pq} = \frac{1}{V_d}\mathcal{A}^{-1}(p^2)B_{pq} - \frac{1}{V_d^2}\mathcal{A}^{-1}(p^2)\int' \frac{V_d d\omega_P P^{d-1}dP}{(2\pi)^d}\chi_{p,-P}\mathcal{A}^{-1}(P^2)B_{Pq} \tag{A.13}$$

can be evaluated by performing the integral over the infinitesimal momentum shell as discussed above. Substituting the result in the r.h.s. of Eq. (A.8) and comparing the corresponding terms on its both sides, one gets:

$$k\partial_k V^{(1)}(\varphi_0) = -\hbar k^d \alpha_d \mathcal{B}\mathcal{A}^{-1}(k^2), \tag{A.14}$$

$$k\partial_k Z_{(0|1)}(k) = \hbar k^d \alpha_d \mathcal{B}_0 \mathcal{A}_0^{-2} Z_{(0|2)}(k), \tag{A.15}$$

$$k\partial_k Z_{(0|2)}(k) = \hbar k^d \alpha_d \left[\frac{V^{(4)}(0)}{\mathcal{A}_0^2} - \frac{2\mathcal{B}^2}{\mathcal{A}_0^3}\right] Z_{(0|2)}(k). \tag{A.16}$$

Here Eq. (A.14) is the first derivative of Eq. (A.11), $\mathcal{A}_0 = \mathcal{A}(k^2)\big|_{\varphi_0=0}$, $\mathcal{B}_0 = \mathcal{B}|_{\varphi_0=0}$.

$\underline{\sigma_2:}$ The trace of the first matrix on the r.h.s. of Eq. (A.9) can be evaluated analogously to that of $A^{-1}B$:

$$-\hbar k^d \alpha_d \int \frac{d\omega_n}{\Omega_d}\frac{V^{(4)}(\varphi_0)}{\mathcal{A}(k^2)}\int d^d z \frac{1}{2}\left[\eta^2(z) + Z_{(0|2)}(z)\Box\eta^2(z)\right]. \tag{A.17}$$

Up to the first order of χ_{pq}, one can write

$$\begin{aligned}-\frac{1}{2}\left(A^{-1}BA^{-1}B\right)_{pq} &= -\frac{1}{2V_d^2}\sum_P{}' \frac{B_{p,-P}B_{Pq}}{\mathcal{A}(p^2)\mathcal{A}(P^2)} + \frac{1}{2V_d^3}\sum_P{}'\sum_Q{}' \frac{B_{p,-P}\chi_{P,-Q}B_{Qq}}{\mathcal{A}(p^2)\mathcal{A}(P^2)\mathcal{A}(Q^2)} \\ &\quad + \frac{1}{2V_d^3}\sum_P{}'\sum_Q{}' \frac{\chi_{p,-Q}B_{Q,-P}B_{Pq}}{\mathcal{A}(p^2)\mathcal{A}(P^2)\mathcal{A}(Q^2)}.\end{aligned} \tag{A.18}$$

The second and third terms give identical contributions to the r.h.s. of Eq. (A.9),

$$\hbar k^d \alpha_d \frac{1}{2}\int \frac{d\omega_n}{\Omega_d}\frac{\mathcal{B}^2}{\mathcal{A}^3(k^2)}\int d^d z Z_{(0|2)}(z)\Box\eta^2(z), \tag{A.19}$$

where only the terms of second order in the gradient are retained. The first term leads to the contribution

$$\frac{\hbar k^d \alpha_d}{2}\int \frac{d^d p_1}{(2\pi)^d}\frac{\mathcal{B}^2}{\mathcal{A}(k^2)\mathcal{A}((kn+p_1)^2)}\eta_{p_1}\eta_{-p_1} \tag{A.20}$$

after performing $\sum_P'$. Expanding the integrand in powers of p_1^μ, we find for the contribution (A.20):

$$\hbar k^d \alpha_d \frac{1}{2}\int \frac{d\omega_n}{\Omega_d}\left\{\frac{\mathcal{B}_0^2}{\mathcal{A}^2(k^2)}\int d^d z \eta^2(z) - \mathcal{G}_{\rho\sigma}\int d^d z \eta(z)\partial^\rho\partial^\sigma\eta(z)\right\}, \tag{A.21}$$

where

$$\mathcal{G}_{\rho\sigma} = \frac{\mathcal{B}_0^2 Z_{(1|1)}(k)}{\mathcal{A}^3(k^2)}\left[\frac{4k^2}{\mathcal{A}(k^2)}Z_{(1|1)}(k)n_\rho n_\sigma - g_{\rho\sigma}\right]. \tag{A.22}$$

Adding all the contributions on the r.h.s. of Eq. (A.9), we can identify the coefficients of the various composite operators on its both sides. Removing now the inhomogeneity of the couplings, we find:

$$k\partial_k V^{(2)}(\varphi_0) = -\hbar k^d \alpha_d \partial_{\varphi_0}^2 \ln \mathcal{A}(k^2), \tag{A.23}$$

$$k\partial_k Z_{(1|1)}(k) = \hbar k^d \alpha_d \frac{1}{d}\mathcal{G}_\mu^\mu(0), \tag{A.24}$$

$$k\partial_k Z_{(0|2)}(k) = \hbar k^d \alpha_d \left[\frac{V^{(4)}(0)}{\mathcal{A}_0^2} - \frac{2\mathcal{B}^2}{\mathcal{A}_0^3}\right] Z_{(0|2)}(k) \tag{A.25}$$

with $\mathcal{G}_\mu^\mu(0) = \mathcal{G}_\mu^\mu\big|_{\varphi_0=0}$. Eq. (A.23) can also be obtained by differentiating both sides of Eq. (A.11) w.r.t. φ_0 two times. Eq. (A.25) is equivalent with Eq. (A.16), whereas Eq. (A.24) is an independent equation.

APPENDIX B: SOLUTION OF RG EQUATIONS FOR THE BLOCKED COUPLINGS IN IMA

We rewrite Eqs. (A.11), (A.15), (A.16), and (A.24) in integral form and substitute the bare values $g_n(\Lambda)$ for the blocked couplings $g_n(k)$ in the integrands. In terms of the integrals I_0 and I_{nrs} given in Appendix E, we have

$$V_k(\varphi_0) = V_\Lambda(\varphi_0) + \hbar\alpha_d I_0, \tag{B.1}$$

$$Z_{(0|2)}(k) = Z_{(0|2)}(\Lambda) - \hbar\alpha_d \left(g_4(\Lambda) I_{002} - 2I_{023}\right) Z_{(0|2)}(\Lambda), \tag{B.2}$$

$$Z_{(0|1)}(k) = Z_{(0|1)}(\Lambda) - \hbar\alpha_d I_{012} Z_{(0|2)}(\Lambda), \tag{B.3}$$

$$Z_{(1|1)}(k) = Z_{(1|1)}(\Lambda)\left[1 - \hbar\alpha_d\left(\frac{4Z_{(1|1)}(\Lambda)}{d} I_{124} - I_{023}\right)\right]. \tag{B.4}$$

Let us evaluate the elements s_{nm} of the operator mixing matrix for the ϕ^3 theory.

From monomials to monomials: We have from Eq. (B.1)

$$g_\ell(k) = g_\ell(\Lambda) + \hbar\alpha_d I_0^{(\ell)}(0), \tag{B.5}$$

and, for the mixing coefficients of the monomial operators,

$$s_{\ell\ell'} = \frac{\partial g_\ell(k)}{\partial g_{\ell'}(\Lambda)} = \delta_{\ell\ell'} + \hbar\alpha_d \frac{\partial I_0^{(\ell)}(0)}{\partial g_{\ell'}(\Lambda)}. \tag{B.6}$$

The partial derivatives on the r.h.s. of Eq. (B.5) can be evaluated by taking the partial derivatives of I_0 (given in Appendix E) w.r.t. φ_0 at $\varphi_0 = 0$ and then differentiating them w.r.t. the appropriate $g_{\ell'}(\Lambda)$, before the integration over κ is performed. Finally, the initial values of the couplings specifying the ϕ^3 theory should be inserted.

In Table IV we summarize the non-trivial operator mixing coefficients in terms of the integrals given in Appendix E and their limiting values for $k = 0$, $\Lambda^2 \gg m^2$ after carrying out the loop integral in IMA. The functions f_a, $a = 0, 2, 4$ in the last column are defined as

$$\begin{aligned}
f_0(\Lambda) &= \left[\ln\left(\frac{\Lambda^2}{m^2} + 1\right) - \frac{3}{2}\right], \\
f_2(\Lambda) &= \left[m^2 \ln\left(\frac{\Lambda^2}{m^2} + 1\right) - \frac{1}{2}\Lambda^2 - \frac{1}{2}m^2\right], \\
f_4(\Lambda) &= \left[m^4 \ln\left(\frac{\Lambda^2}{m^2} + 1\right) + \frac{1}{2}\Lambda^4 - \Lambda^2 m^2\right], \\
f_6(\Lambda) &= -\frac{1}{3}\Lambda^6 + \frac{1}{2}\Lambda^4 m^2 - \Lambda^2 m^4 + m^6 \ln\left(\frac{\Lambda^2}{m^2} + 1\right).
\end{aligned} \tag{B.7}$$

The theory is renormalizable and we see that monomial operators $\varphi^{\ell'}$ of dimension not greater than the dimension of the monomial φ^ℓ are the only ones admixing to the blocked operator $\{\varphi^\ell\}_{k\to 0}$ with UV divergent coefficients. Were the theory non-renormalizable, e.g. a ϕ^4 term with non-vanishing coupling $g_4(\Lambda) \neq 0$ included, then also the admixture of higher dimensional operators would occur with UV divergent coefficients.

These results can be compared with the one-loop perturbative result given in [6] (p.145, (6.2.11)) for the renormalized operator $\frac{1}{2}[\phi^2]_R$, replacing $2/(d-6)$ by $-\ln(\Lambda^2/m^2)$ and using $\alpha_6 = 1/(128\pi^3)$ (in our notations):

$$\frac{1}{2}[\phi^2]_R = M_1(\phi)\left(Z^{-1}\right)_{12} + M_2(\phi)\left(Z^{-1}\right)_{22} + D_{(0|1)}(\phi)\left(Z^{-1}\right)_{(0|1)2} + \dots \tag{B.8}$$

where ... stands for higher order terms and

$$\left(Z^{-1}\right)_{12} = -\hbar\frac{\lambda m^2}{128\pi^3}\ln\frac{\Lambda^2}{m^2}, \qquad \left(Z^{-1}\right)_{22} = 1 - \hbar\frac{\lambda^2}{128\pi^3}\ln\frac{\Lambda^2}{m^2}, \qquad \left(Z^{-1}\right)_{(0|1)2} = \hbar\frac{\lambda}{6\cdot 128\pi^3}\ln\frac{\Lambda^2}{m^2}. \tag{B.9}$$

As far as the mixing of monomials is considered, this illustrates that the operator mixing matrix s_{nm} in the IMA in the limit $k \to 0$ and $\Lambda^2 \gg m^2$ satisfies the relation $s_{\ell\ell'} = Z_{\ell\ell'}$, where Z_{nm} is the one-loop perturbative operator mixing matrix.

From monomials to derivative operators: Eq. (B.5) obtained from (B.1) can also be used to determine the admixture of monomials to the blocked derivative operators in the IMA. Since $I_0^{(n)}(0)$ depends only on $z = Z_{(1|1)}(\Lambda)$ and not on the couplings of the other derivative operators, one obtains $s_{\ell(r|s)}(k) = 0$ for $(r|s) \neq (1|1)$, and

$$s_{\ell(1|1)}(k) = \hbar\alpha_d \frac{\partial I_0^{(n)}(0)}{\partial z}. \tag{B.10}$$

The first few non-vanishing operator mixing coefficients and their limiting values for $k = 0$, $\Lambda^2 \gg m^2$ are collected in Table V. Thus, we can write for the blocked operator (with constant argument!): $\{D_{(1|1)}(\varphi_0)\}_k = \sum_{n=0}^{\infty} s_{n(1|1)}(k)\varphi_0^n/n!$. Again, the contributions of the derivative operators to the blocked derivative operator $\{D_{(1|1)}(\varphi_k)\}_k$ can only be seen if the latter is considered at the general argument $\varphi_k(x)$.

From derivative operators to monomials and derivative operators: Differentiating the solutions (B.4), (B.2), and (B.3) w.r.t. the various bare couplings the following additional (non-vanishing) operator mixing coefficients and their limiting values for $k = 0$, $\Lambda^2 \gg m^2$ are shown in Table VI.

The result obtained for $Z_{(1|1)}(k = 0)$ in the IMA is in agreement with the one-loop perturbative result for Z_ϕ:

$$Z_\phi = 1 + \frac{\hbar\lambda^2}{12 \cdot 64\pi^3}\left(\ln\frac{\Lambda^2}{m^2} - \frac{5}{6}\right) \tag{B.11}$$

(see [6], p.58) as expected, since the correspondence $Z_{(1|1)}(k = 0) = Z_\phi$ should hold due to

$$\int d^dx \frac{1}{2} Z_\phi (\partial_\mu\phi)^2 = \int d^dx \frac{1}{2} Z_\phi (2D_{(1|1)} - D_{(0|2)}) = \int d^dx Z_{(1|1)} D_{(1|1)}. \tag{B.12}$$

Using $Z_{(0|2)}(\Lambda) = -\frac{1}{2}$ (owing to Eq. (7.4)) and $Z_{(0|1)}(\Lambda) = 0$, a comparison of $s_{(0|1)2}$ with the corresponding one-loop perturbative result (see relation (B.8) taken from [6]) gives $s_{(0|1)2} = 3Z_{(0|1)2}$. Since $s_{(0|1)2}$ obtained above agrees with our perturbative one-loop result in Appendix D, so that both results contradict to (B.8), there ought to be a misprint in [6].

APPENDIX C: CONNECTION MATRIX

First, we read off the beta-functions from the right hand sides of Eqs. (A.11), (A.15), (A.16), and (A.24). The beta-functions for the couplings of the monomials are given by $\beta_\ell(k) = \hbar\alpha_6 k^6\, \partial_{\varphi_0}^\ell \ln \mathcal{A}(k^2)\big|_{\varphi_0=0}$, whereas the other beta-functions can be read off directly. With the notations $m^2(k) = V^{(2)}(0)$, $\lambda(k) = V^{(3)}(0)$, $g(k) = V^{(4)}(0)$, $g_\ell(k) = V^{(\ell>4)}(0)$, and $G = \left[Z_{(1|1)}(k)k^2 + m^2(k)\right]^{-1}$, the explicit forms of the beta-functions are given in Table VII.

The elements of the connection matrix, shown in Table VIII are obtained by differentiation according to the second equation of (4.14).

APPENDIX D: PERTURBATIVE ONE-LOOP RESULTS FOR OPERATOR MIXING

Here we determine the operator mixing matrix at one-loop order perturbatively for $\lambda\phi^3$ theory in dimension $d = 6$. Let us decompose the action S into the free part

$$S_0 = \int d^dx \left[-\frac{1}{2} Z_R \phi\Box\phi + \frac{1}{2} m_R^2 \phi^2\right], \tag{D.1}$$

the interaction part

$$S_I = \int d^dx \left[-\frac{1}{2} g_{(1|1)R}(x)\phi\Box\phi + \frac{1}{2} g_{2R}(x)\phi^2 + \sum_{n\neq(1|1),2} G_{nR}(x) O_n(\phi(x))\right], \tag{D.2}$$

and the counterterms

$$S_{c.t.} = \int d^dx \sum_n c_n(g_{mR}) O_n(\phi(x)). \tag{D.3}$$

Here the same base operators are taken into account as in the blocked action (7.3). The one-loop effective action is given by

$$
\begin{aligned}
\Gamma_1 &= \frac{\hbar}{2}\ \mathrm{Tr}\ \ln\left(1_{pq} + G_R(p^2)(S_I^{(2)})_{-p\ q}\right) \\
&= -\frac{\hbar}{2}\sum_{v=1}^{\infty}\frac{(-1)^v}{v}\int_{p_1}\cdots\int_{p_v} G_R(p_1^2)(S_I^{(2)})_{-p_1\ p_2}\cdots G_R(p_v^2)(S_I^{(2)})_{-p_v\ p_1}
\end{aligned}
\tag{D.4}
$$

with $G_R(p^2) = (Z_R p^2 + m_R^2)^{-1}$ and the second functional derivative of the interaction piece of the action:

$$
\begin{aligned}
(S_I^{(2)})_{pq} = \int dx e^{i(p+q)x} \Bigg[& g_{2R}(x) + g_{(1|1)R}(x)\frac{1}{2}(p^2+q^2) \\
& + \sum_{\ell=3}^{\infty}(g_{\ell R} + g_{\ell R}(x))\,\frac{1}{(\ell-2)!}\phi^{\ell-2}(x) + \left(Z_{(0|2)R} + g_{(0|2)R}(x)\right)(p+q)^2\Bigg].
\end{aligned}
\tag{D.5}
$$

In the second order of the derivative expansion, the counterterms are defined through the relations:

$$
\begin{aligned}
-\left.\frac{\delta\Gamma_1}{\delta\varphi(p)}\right|_{\phi,g_{\bar n R}=0} &= \left(c_1 + c_{(0|1)}p^2 + o(p^4)\right)(2\pi)^d\delta(p), \\
-\left.\frac{\delta^2\Gamma_1}{\delta\varphi(p_1)\delta\varphi(p_2)}\right|_{\phi,g_{\bar n R}=0} &= \left(c_2 + c_{(1|1)}p_1^2 + o(p_1^4)\right)(2\pi)^d\delta(p_1+p_2), \\
-\left.\frac{\delta^s\Gamma_1}{\delta\varphi(p_1)\cdots\delta\varphi(p_s)}\right|_{\phi,g_{\bar n R}=0,p_i^2=0} &= c_s(2\pi)^d\delta(p_1+\ldots+p_s), \qquad (s\geq 3) \\
-Z_{(0|2)R}\left.\frac{\delta^3\Gamma_1}{\delta\varphi(p_1)\delta\varphi(p_2)\delta g_{(0|2)R}(x)}\right|_{\phi,g_{\bar n R}=0} &= c_{(0|2)}(p_1+p_2)^2 e^{i(p_1+p_2)x}.
\end{aligned}
\tag{D.6}
$$

A straightforward evaluation results in the expressions for the counterterms given in Table IX.

For the elements of the operator mixing matrix (6.11) we obtain, with $Z_R \equiv Z_{(1|1)R} = 1$, $Z_{(0|2)R} = -\frac{1}{2}$, $g_{nR} = 0$ for $n \neq 2, (1|1), (0|2), 3$, the expressions shown in Table X.

APPENDIX E: PARTICULAR INTEGRALS

In order to obtain the explicit forms of the solutions of Eqs. (B.1), (B.4), (B.3), and (B.2) the following integrals are needed:

$$
\begin{aligned}
I_0 &= \int_k^{\Lambda} dp p^{d-1}\ln\left(zp^2 + V_\Lambda^{(2)}(\varphi_0)\right), \\
I_{nrs} &= \lambda_\Lambda^r\int_k^{\Lambda} dp p^{d-1}\,p^{2n}G_\Lambda^s = \frac{\lambda_\Lambda^r}{2z^s}\sum_{i=0}^{j+n-1}\binom{j+n-1}{i}\left(-\frac{m_\Lambda^2}{z}\right)^{j+n-1-i}\int_{k^2}^{\Lambda^2} du\left(u+\frac{m_\Lambda^2}{z}\right)^{i-s}
\end{aligned}
\tag{E.1}
$$

for even dimensions $d = 2j$, where $G_\Lambda = \left[zp^2 + m_\Lambda^2\right]^{-1}$. For dimensions $d = 6$ we obtain:

$$
I_0 = \frac{1}{6}\left[u^3\ln\left(zu + V_\Lambda^{(2)}(\varphi_0)\right) - \frac{1}{3}(u+c)^3 + \frac{3}{2}c(u+c)^2 - 3c^2u + c^3\ln(u+c)\right]_{u=k^2}^{u=\Lambda^2}
\tag{E.2}
$$

with $c = V_\Lambda^{(2)}(\varphi_0)/z$. The evaluation of the integrals I_{nrs} is straightforward.

TABLE I. Comparison of various notions in the renormalization group (RG) method and the renormalized perturbation expansion (RPE)

RG	RPE
coupling at UV scale: $g_n(\Lambda)$ blocked coupling: $g_n(k)$	bare coupling: g_n renormalized coupling: g_{nR}
operator mixing matrix: $s_{nm}(k,\Lambda)$ defined as $s_{nm} = \frac{\partial g_n(k)}{\partial g_m(\Lambda)}$	operator mixing matrix: Z_{nm} defined via $\left(Z^{-1}\right)_{mn} = \frac{\partial g_m}{\partial g_{nR}}$
operator at UV scale: $O_{\tilde{n}}(\phi)$	bare operator: $O_{\tilde{n}}(\phi)$
blocked operator: $\{O_{\tilde{n}}(\varphi_k)\}_k = \frac{\delta S_k}{\delta G_{\tilde{n}}(\Lambda)}\Big\vert_{g_{\tilde{m}}(\Lambda)=0}$ operator mixing: $\{O_{\tilde{n}}(\varphi_k)\}_k = \sum_j O_{\tilde{m}}(\varphi_k) s_{mn}(k)$	not used
operator obtained by inverse blocking (if exists): $[O_{\tilde{n}}(\phi)]_k = \sum_m O_{\tilde{m}}(\phi)(s^{-1})_{mn}(k,\Lambda)$	renormalized operator: $[O_{\tilde{n}}(\phi)]_R = \sum_m O_{\tilde{m}}(\phi)(Z^{-1})_{mn}$

TABLE II. Orders of magnitude of the operator mixing coefficients s_{nm} in IMA for $\lambda\phi^3$ theory for $k=0$, $\Lambda^2 \gg m^2$. $E(\Lambda)$ indicates a constant plus logarithmically divergent one-loop contributions; $M_n = \varphi^n/n!$; d_n, d_m stand for the momentum dimensions of the operators $\{O_{\tilde{n}}\}$, $O_{\tilde{m}}$, resp.

	d_m	0	2	4		6			8
d_n	$\{O_{\tilde{n}}\} \setminus O_{\tilde{m}}$	M_0	M_1	M_2	$D_{(0\vert 1)}$	M_3	$D_{(0\vert 2)}$	$D_{(1\vert 1)}$	M_4
0	$\{M_0\}$	1	0	Λ^4	0	0	0	Λ^6	0
2	$\{M_1\}$	0	1	Λ^2	0	Λ^4	0	Λ^4	0
4	$\{M_2\}$	0	0	$E(\Lambda)$	0	Λ^2	0	Λ^2	Λ^4
	$\{D_{(0\vert 1)}\}$	0	0	$\ln\Lambda$	1	Λ^2	Λ^2	Λ^2	0
6	$\{M_3\}$	0	0	Λ^0	0	$E(\Lambda)$	0	$\ln\Lambda$	Λ^2
	$\{D_{(0\vert 2)}\}$	0	0	Λ^0	0	$\ln\Lambda$	$E(\Lambda)$	$\ln\Lambda$	Λ^2
	$\{D_{(1\vert 1)}\}$	0	0	Λ^0	0	$\ln\Lambda$	0	$E(\Lambda)$	0
8	$\{M_4\}$	0	0	Λ^0	0	Λ^0	0	Λ^0	$E(\Lambda)$

TABLE III. Elements of the connection matrix $\gamma_{nm}(k)$ for a theory with polynomial potential; the non-trivial matrix elements are indicated by $\times$, d_n, d_m denote the momentum dimensions of the base operators $O_{\tilde{n}}(\phi)$, $O_{\tilde{m}}(\phi)$, resp.

	d_m	0	2	4		6			8
d_n	$n \setminus m$	0	1	2	(0|1)	3	(0|2)	(1|1)	4
0	0	0	0	$\times$	0	0	0	$\times$	0
2	1	0	0	$\times$	0	$\times$	0	$\times$	0
4	2	0	0	$\times$	0	$\times$	0	$\times$	$\times$
	(0|1)	0	0	$\times$	0	$\times$	$\times$	$\times$	0
6	3	0	0	$\times$	0	$\times$	0	$\times$	$\times$
	(0|2)	0	0	$\times$	0	$\times$	$\times$	$\times$	$\times$
	(1|1)	0	0	$\times$	0	$\times$	0	$\times$	0
8	4	0	0	$\times$	0	$\times$	0	$\times$	$\times$

TABLE IV. The mixing matrix elements $s^{IMA}_{mn}(k)$.

m, n	integral form	after integration
0, 2	$\hbar\alpha_6\partial_{m^2}I_0(0) = \hbar\alpha_6 I_{001}$	$\hbar\alpha_6 f_4(\Lambda)/2$
1, 2	$\hbar\alpha_6\partial_{m^2}I_0^{(1)}(0) = -\hbar\alpha_6 I_{012}$	$\hbar\alpha_6\lambda f_2(\Lambda)$
2, 2	$1+\hbar\alpha_6\partial_{m^2}I_0^{(2)}(0) = 1+2\hbar\alpha_6 I_{023}$	$1+\hbar\alpha_6\lambda^2 f_0(\Lambda)$
3, 2	$\hbar\alpha_6\partial_{m^2}I_0^{(3)}(0) = -6\hbar\alpha_6 I_{034}$	$-\hbar\alpha_6\lambda^3/m^2$
4, 2	$\hbar\alpha_6\partial_{m^2}I_0^{(4)}(0) = 24\hbar\alpha_6 I_{045}$	$\hbar\alpha_6\lambda^4/m^4$
1, 3	$\hbar\alpha_6\partial_\lambda I_0^{(1)}(0) = \hbar\alpha_6 I_{001}$	$\hbar\alpha_6 f_4(\Lambda)/2$
2, 3	$\hbar\alpha_6\partial_\lambda I_0^{(2)}(0) = -2\hbar\alpha_6 I_{012}$	$2\hbar\alpha_6\lambda f_2(\Lambda)$
3, 3	$1+\hbar\alpha_6\partial_\lambda I_0^{(3)}(0) = 1+6\hbar\alpha_6 I_{023}$	$1+3\hbar\alpha_6\lambda^2 f_0(\Lambda)$
4, 3	$\hbar\alpha_6\partial_\lambda I_0^{(4)}(0) = -24\hbar\alpha_6 I_{034}$	$-4\hbar\alpha_6\lambda^3/m^2$
2, 4	$\hbar\alpha_6\partial_g I_0^{(2)}(0) = \hbar\alpha_6 I_{001}$	$\hbar\alpha_6 f_4(\Lambda)/2$
3, 4	$\hbar\alpha_6\partial_g I_0^{(3)}(0) = -3\hbar\alpha_6 I_{012}$	$3\hbar\alpha_6\lambda f_2(\Lambda)$
4, 4	$1+\hbar\alpha_6\partial_g I_0^{(4)}(0) = 1+12\hbar\alpha_6 I_{023}$	$1+6\hbar\alpha_6\lambda^2 f_0(\Lambda)$

TABLE V. The mixing matrix elements $s^{IMA}_{\ell(m|n)}(k)/\hbar\alpha_6$.

$\ell(m\|n)$	integral form	after integration
0(1|1)	I_{101}	$-f_6(\Lambda)/2$
1(1|1)	$-I_{112}$	$-\lambda[3f_4(\Lambda)-\Lambda^4+\Lambda^2m^2-m^4]/6$
2(1|1)	$2I_{123}$	$-\lambda^2[6f_2(\Lambda)+\Lambda^2-2m^2]$
3(1|1)	$-6I_{134}$	$-\lambda^3[3f_0(\Lambda)-1]$
4(1|1)	$24I_{145}$	$3\lambda^4/m^2$

TABLE VI. The mixing matrix elements $s^{IMA}_{index}(k)$.

index	integral form	after integration
(1|1)2	$\hbar\alpha_6(8Z^2_{(1\|1)}I_{125} - 9Z_{(1\|1)}I_{024})/3$	$-\hbar\alpha_6\lambda^2/6m^2$
(1|1)3	$-\hbar\alpha_6(4Z^2_{(1\|1)}I_{114} - 6Z_{(1\|1)}I_{013})/3$	$\hbar\alpha_6\lambda[3f_0(\Lambda)+2]/9$
(1|1)(1|1)	$1-\hbar\alpha_6(13Z_{(1\|1)}I_{124} - 3I_{023} - 8Z^2_{(1\|1)}I_{225})/3$	$1-\hbar\alpha_6\lambda^2[6f_0(\Lambda)+1]/18$
(0|2)2	$-6\hbar\alpha_6 Z_{(0\|2)}(\Lambda)I_{024}$	$-\hbar\alpha_6\lambda^2 Z_{(0\|2)}(\Lambda)/m^2$
(0|2)3	$4\hbar\alpha_6 Z_{(0\|2)}(\Lambda)I_{013}$	$2\hbar\alpha_6\lambda Z_{(0\|2)}(\Lambda)f_0(\Lambda)$
(0|2)4	$-\hbar\alpha_6 Z_{(0\|2)}(\Lambda)I_{002}$	$\hbar\alpha_6 Z_{(0\|2)}(\Lambda)f_2(\Lambda)$
(0|2)(0|2)	$1+2\hbar\alpha_6 I_{023}$	$1+\hbar\alpha_6\lambda^2 f_0(\Lambda)$
(0|2)(1|1)	$-6\hbar\alpha_6 Z_{(0\|2)}(\Lambda)I_{124}$	$-3\hbar\alpha_6\lambda^2 Z_{(0\|2)}(\Lambda)[3f_0(\Lambda)+2]/3$
(0|1)2	$2\hbar\alpha_6 Z_{(0\|2)}(\Lambda)I_{013}$	$\hbar\alpha_6\lambda Z_{(0\|2)}(\Lambda)f_0(\Lambda)$
(0|1)3	$-\hbar\alpha_6 Z_{(0\|2)}(\Lambda)I_{002}$	$\hbar\alpha_6 Z_{(0\|2)}(\Lambda)f_2(\Lambda)$
(0|1)(0|1)	1	1
(0|1)(0|2)	$-\hbar\alpha_6 I_{012}$	$\hbar\alpha_6\lambda f_2(\Lambda)$
(0|1)(1|1)	$2\hbar\alpha_6 Z_{(0\|2)}(\Lambda)I_{113}$	$-\hbar\alpha_6\lambda Z_{(0\|2)}(\Lambda)[6f_2(\Lambda)+\Lambda^2-2m^2]/2$

TABLE VII. The beta-functions $\beta_{index}(k)/\hbar\alpha_6 k^6$.

index	beta-function
0	$-\ln G$
1	$G\lambda$
2	$G(g-G\lambda^2)$
(0|1)	$G^2\lambda Z_{(0\|2)}$
3	$G(g_5 - 3gG\lambda + 2G^2\lambda^3)$
(0|2)	$G^2 Z_{(0\|2)}(g-2G\lambda)$
(1|1)	$G^3\lambda^2 Z_{(1\|1)}(2k^2 G Z_{(1\|1)} - 3)/3$
4	$G(g_6 - 4g_5 G\lambda - 3g^2 G + 12gG^2\lambda^2 - 6G^3\lambda^4)$

TABLE VIII. The non-vanishing connection matrix elements $k\gamma_{index}(k)/\hbar\alpha_6 k^6$.

index	connection matrix
02	G
0(1|1)	$k^2 G$
12	$-G^2\lambda$
13	G
1(1|1)	$-k^2 G^2 \lambda$
22	$G^2(-g + 2G\lambda^2)$
23	$-2G^2\lambda$
2(1|1)	$k^2 G^2(-g + 2G\lambda^2)$
24	G
(0|1)2	$-2G^3\lambda Z_{(0\|2)}$
(0|1)3	$G^2 Z_{(0\|2)}$
(0|1)	$G^2\lambda$
(0|1)(1|1)	$-2k^2 G^3 \lambda Z_{(0\|2)}$
32	$-G^2(g_5 - 6gG\lambda + 6G^2\lambda^3)$
33	$-G^2(3g - 6G\lambda^2)$
3(1|1)	$-k^2 G^2(g_5 - 6gG\lambda + 6G^2\lambda^3)$
34	$-3G^2\lambda$
(0|2)2	$2G^3 Z_{(0\|2)}(g + 3G\lambda^2)$
(0|2)3	$-4G^3\lambda Z_{(0\|2)}$
(0|2)(0|2)	$G^2(g - 2G\lambda^2)$
(0|2)(1|1)	$-2k^2 G^3 Z_{(0\|2)}(g - 3G\lambda^2)$
(0|2)4	$G^2 Z_{(0\|2)}$
(1|1)2	$G^4\lambda^2 Z_{(1\|1)}(9 - 8k^2 G Z_{(1\|1)})/3$
(1|1)3	$G^3\lambda Z_{(1\|1)}(4k^2 G Z_{(1\|1)} - 6)/3$
(1|1)(1|1)	$G^3\lambda^2(12k^2 G Z_{(1\|1)} - 3 - 8k^4 G^2 Z^2_{(1\|1)})/3$
42	$G^2(-g_6 + 8g_5 G\lambda + 6g^2 G - 36gG^2\lambda^2 + 24G^3\lambda^4)$
43	$-4G^2(g_5 - 6gG\lambda + 6G^2\lambda^3)$
4(1|1)	$-k^2 G^2(g_6 - 8\lambda g_5 G\lambda - 6g^2 G + 36gG^2\lambda^2 - 24G^3\lambda^4)$
44	$-6G^2(g - 2G\lambda^2)$

TABLE IX. The counterterms expressed in terms of the integrals with $\lambda = g_{3R}$.

index	$c_{index}/\hbar\alpha_d$
02	G
1	$-I_{011}$
2	$I_{022} - g_{4R} I_{001}$
(0|1)	$Z_{(0\|2)R} I_{012}$
3	$3g_{4R} I_{012} - 2I_{033}$
(0|2)	$(g_{4R} I_{002} - 2I_{023})\, Z_{(0\|2)R}$
(1|1)	$\frac{4Z_R^2}{d} I_{124} - Z_R I_{023}$
4	$3g_{4R}^2 I_{002} - 12 g_{4R} I_{023} + 6I_{044}$

TABLE X. The operator mixing matrix $(Z^{-1})_{index}$.

index	mixing matrix
10, 11	1
12	$\hbar\alpha_d I_{012}$
1(0|1)	0
13	$-\hbar\alpha_d I_{001}$
1(0|2)	0
1(1|1)	$\hbar\alpha_d I_{112}$
14	0
20, 21	0
22	$1+\hbar\alpha_d\left(g_{4R}I_{002}-2I_{023}\right)$
2(0|1)	0
23	$2\hbar\alpha_d I_{012}$
2(0|2)	0
2(1|1)	$\hbar\alpha_d\left(g_{4R}I_{102}-2I_{123}\right)$
24	$-\hbar\alpha_d I_{001}$
(0|1)0, (0|1)1	0
(0|1)2	$-2\hbar\alpha_d I_{013}Z_{(0\|2)R}$
(0|1)(0|1)	1
(0|1)3	$\hbar\alpha_d I_{002}Z_{(0\|2)R}$
(0|1)(0|2)	$\hbar\alpha_d I_{012}$
(0|1)(1|1)	$-2\hbar\alpha_d I_{113}Z_{(0\|2)R}$
(0|1)4	0
30, 31	0
32	$6\hbar\alpha_d\left(I_{034}-g_{4R}I_{013}\right)$
3(0|1)	0
33	$1+3\hbar\alpha_d\left(g_{4R}I_{002}-2I_{023}\right)$
3(0|2)	0
3(1|1)	$6\hbar\alpha_d\left(I_{134}-g_{4R}I_{113}\right)$
34	$3\hbar\alpha_d I_{012}$
(0|2)0, (0|2)1	0
(0|2)2	$\hbar\alpha_d\left(6I_{024}-2g_{4R}I_{003}\right)Z_{(0\|2)R}$
(0|2)(0|1)	0
(0|2)3	$-4\hbar\alpha_d I_{013}Z_{(0\|2)R}$
(0|2)(0|2)	$1+\hbar\alpha_d\left(g_{4R}I_{002}-2I_{023}\right)$
(0|2)(1|1)	$\hbar\alpha_d\left(6I_{124}-2g_{4R}I_{103}\right)Z_{(0\|2)R}$
(0|2)4	$\hbar\alpha_d I_{002}Z_{(0\|2)R}$
(1|1)0, (1|1)1	0
(1|1)2	$\hbar\alpha_d\left(3dZ_RI_{024}-16Z_R^2I_{125}\right)/d$
(1|1)(0|1)	0
(1|1)3	$\hbar\alpha_d\left(8Z_R^2I_{114}-2dZ_RI_{013}\right)/d$
(1|1)(0|2)	0
(1|1)(1|1)	$1+\hbar\alpha_d\left[-16Z_R^2I_{225}+(8+3d)\,Z_RI_{124}-dI_{023}\right]/d$
(1|1)4	0
40, 41	0
42	$6\hbar\alpha_d\left(-g_{4R}^2I_{003}+6g_{4R}I_{024}-4I_{045}\right)$
4(0|1)	0
43	$24\hbar\alpha_d\left(I_{034}-g_{4R}I_{013}\right)$
4(0|2)	0
4(1|1)	$6\hbar\alpha_d\left(-g_{4R}^2I_{103}+6g_{4R}I_{124}-4I_{145}\right)$
44	$1+6\hbar\alpha_d\left(g_{4R}I_{002}-2I_{023}\right)$